Classification of the Ricci tensor in space-times admitting a four-parameter group of motions acting on non-null hypersurfaces

1980 J. Phys. A: Math. Gen. 13 1709

(http://iopscience.iop.org/0305-4470/13/5/028)

J. Phys. A: Math. Gen. 13 (1980) 1709–1719. Printed in Great Britain

Classification of the Ricci tensor in space–times admitting a four-parameter group of motions acting on non-null hypersurfaces

M Abdel-Megied
Department of Mathematics, University of Riyad, Kingdom of Saudi Arabia

Received 12 January 1978, in final form 3 January 1979

Abstract. The Plebanski type of the Ricci tensor for all metrics admitting a four-parameter group of motions acting on non-null (space-like or time-like) hypersurfaces is determined by using the Newman–Penrose formalism and a classification method due to Ludwig and Scanlan. For each metric, all possible degeneracies of the Plebanski type are considered.

1. Introduction

The methods of classification of the Ricci tensor in general relativity have been given by many authors (Plebanski 1964, Plebanski and Stachel 1968, Ludwig and Scanlan 1971, Barnes 1974, Hall 1976). The spinor method of Ludwig and Scanlan (1971), here referred to as LS, is based on the classification of the trace-free part of the Ricci tensor. The results of this method will be applied here to classify the Ricci tensor of space–times admitting a G_4 on non-null hypersurfaces. The trace-free components of the Ricci tensor will be obtained from the Newman–Penrose equations (1962), here referred to as NP equations. The calculations are performed by using the complex vectorial formalism (Cahen *et al* 1967) rather than the spin coefficients method. This can be done since the translation of the spin coefficients in terms of the Ricci rotation coefficients is known (Israel 1970). So, the Plebanski type of the Ricci tensor and its possible degeneracies will impose some restrictions on the unknown functions in the metric in question. In this way, one can obtain a class of spatially homogeneous solutions.

Now, by spatial homogeneity of a space–time, we mean that there exists an r-parameter group of motions G_r ($r \geq 3$) acting transitively on space-like hypersurfaces. The orbit of a point x in a space–time, i.e. the set of all points into which it is mapped by the group operations, will be a space-like three-dimensional manifold. The subgroup of G_r which maps a point x into itself is called the isotropy group I_s of x, where s is the number of parameters of the isotropy group. The orbits are isomorphic to the factor manifold G_r/I_s. Since the dimension q of the orbit is connected to r and s by the relation $q = r - s$ (Eisenhart 1961, Ehlers and Kundt 1962), we have $r - s = 3$. A complete survey of the geometry of spatially homogeneous space–times can be found in an exhaustive article (174 references) by MacCallum (1973).

Consider spatially homogeneous space–times admitting a four-parameter group of motions G_4. From the above discussion, we see that the isotropy group must be a one-parameter group. So, the metric of such spaces can be only one of the Petrov types

D, N or O (Ehlers and Kundt 1962), and the nature of the isotropy here is such that the Petrov type of the Weyl tensor must be D or O (Ehlers and Kundt 1962). The nature of the isotropy also puts restrictions on the Segre type of the Ricci tensor, the only allowable types being [11(11)], [2(11)], $[z, \bar{z}(11)]$ (Crade and Hall 1979, MacCallum 1979).

In § 2, we shall calculate the spin coefficients for each type of metric, using the complex vectorial formalism (Cahen *et al* 1967, Israel 1970). In § 3, the NP equations will be used to determine the Petrov type corresponding to each Bianchi type metric and to obtain the components of the trace-free Ricci tensor; then, after a comparison with LS, the Plebanski type can be determined.

Throughout this paper, the signature of the space–time, in a local Lorentz frame, is −2. Greek indices with values 0, 1, 2, 3 are coordinate indices, while Latin letters are tetrad indices.

2. Complex null tetrad and the spin coefficients

All metrics admitting a four-parameter group of motions acting on space-like or time-like hypersurfaces were given by Petrov (1969). Ray and Zimmerman (1977) carried out a systematic investigation of these metrics. They found some exact solutions called space-like dust solutions.

Now, all Bianchi-type metrics which we consider here are:

$$(G_4\,\mathrm{I})\qquad -\mathrm{d}s^2=\mathrm{d}t^2+2A(t)\,\mathrm{d}x^1\,\mathrm{d}x^3+B^2(t)(\mathrm{d}x^2+x^1\,\mathrm{d}x^3)^2,\quad A(t)>0, \tag{2.1}$$

$$(G_4\,\mathrm{III})\qquad \mathrm{d}s^2=\mathrm{d}t^2-A^2(t)(\mathrm{d}x^{1^2}+\mathrm{d}x^{3^2})-B^2(t)(\mathrm{d}x^2+x^1\,\mathrm{d}x^3)^2, \tag{2.2}$$

$$(G_4\,\mathrm{IV})\qquad -\mathrm{d}s^2=\mathrm{d}t^2+A^2(t)\,\mathrm{d}x^{1^2}+2B(t)\,\mathrm{e}^{x^1}\,\mathrm{d}x^2\,\mathrm{d}x^3,\qquad B(t)>0, \tag{2.3}$$

$$(G_4\,\mathrm{V})\qquad \mathrm{d}s^2=\mathrm{d}t^2-A^2(t)\,\mathrm{d}x^{1^2}-B^2(t)\,\mathrm{e}^{2x^1}(\mathrm{d}x^{2^2}+\mathrm{d}x^{3^2}), \tag{2.4}$$

$$(G_4\,\mathrm{VI}_1)\qquad -\mathrm{d}s^2=\mathrm{d}t^2+A^2(t)\,\mathrm{d}x^{1^2}+2B^2(t)\,\mathrm{d}x^2\,\mathrm{d}x^3, \tag{2.5}$$

$$(G_4\,\mathrm{VI}_3)\qquad -\mathrm{d}s^2=\mathrm{d}t^2+A(t)(2\,\mathrm{d}x^1\,\mathrm{d}x^2+\mathrm{d}x^{3^2})+B(t)\,\mathrm{d}x^{2^2},\quad A(t),\,B(t)>0, \tag{2.6}$$

$$(G_4\,\mathrm{VI}_4)\qquad \mathrm{d}s^2=\mathrm{d}t^2-A^2(t)\,\mathrm{d}x^{1^2}-B^2(t)(\mathrm{d}x^{2^2}+\mathrm{d}x^{3^2}), \tag{2.7}$$

$$(G_4\,\mathrm{VII}_2)\qquad \mathrm{d}s^2=-\mathrm{d}t^2+2A^2(t)(\mathrm{d}x^2-x^2\,\mathrm{d}x^3)\,\mathrm{d}x^1+B^2(t)(\mathrm{d}x^2-x^2\,\mathrm{d}x^3)^2-\tfrac{1}{2}A^2(t)\,\mathrm{d}x^{3^2}, \tag{2.8}$$

$$(G_4\,\mathrm{VIII}_1)\qquad \mathrm{d}s^2=\mathrm{d}t^2-A^2(t)\,\mathrm{d}x^{1^2}-B^2(t)(\mathrm{d}x^{2^2}+\sin^2 x^2\,\mathrm{d}x^{3^2}). \tag{2.9}$$

We have excluded the metric $G_4\,\mathrm{VIII}_2$ since it does not have a Lorentz signature. The metric $G_4\,\mathrm{VII}_1$ will be similar to $G_4\,\mathrm{VIII}_1$ when one writes the latter metric with hyperbolic functions instead of the trigonometric functions.

Now we construct a complex null tetrad l^μ, n^μ, m^μ, $\bar{m}^\mu$ (Sachs 1961) for each metric as follows:

$$(G_4\,\mathrm{I})\qquad l^\mu=\delta^\mu_1,\qquad n^\mu=(x^1\delta^\mu_2-\delta^\mu_3)/A,\quad m^\mu,\,\bar{m}^\mu=(\delta^\mu_0\pm\mathrm{i}\delta^\mu_2/B)/\sqrt{2}, \tag{2.10}$$

(G_4 III) $l^\mu = (\delta_0^\mu + \delta_2^\mu/B)/\sqrt{2}$, $n^\mu = (\delta_0^\mu - \delta_2^\mu/B)/\sqrt{2}$,

$$m^\mu, \bar{m}^\mu = (1/\sqrt{2}A)(\delta_1^\mu \mp \mathrm{i}x^1\delta_2^\mu \pm \mathrm{i}\delta_3^\mu) \tag{2.11}$$

(G_4 IV) $l^\mu = \delta_2^\mu$, $n^\mu = (-\mathrm{e}^{-x^1}/B)\delta_3^\mu$, $m^\mu, \bar{m}^\mu = (\delta_0^\mu \pm \mathrm{i}\delta_1^\mu/A)/\sqrt{2}$, (2.12)

(G_4 V) $l^\mu = (\delta_0^\mu + \delta_1^\mu/A)/\sqrt{2}$, $n^\mu = (\delta_0^\mu - \delta_1^\mu/A)/\sqrt{2}$,

$$m^\mu, \bar{m}^\mu = (\mathrm{e}^{-x^1}/\sqrt{2}B)(\delta_2^\mu \pm \mathrm{i}\delta_3^\mu) \tag{2.13}$$

(G_4 VI$_1$) $l^\mu = \delta_2^\mu$, $n^\mu = -\delta_3^\mu/B^2$,

$$m^\mu, \bar{m}^\mu = (\delta_0^\mu \pm \mathrm{i}\delta_1^\mu/A)/\sqrt{2}, \tag{2.14}$$

(G_4 VI$_3$) $l^\mu = \delta_1^\mu$, $n^\mu = (B/2A^2)\delta_1^\mu - \delta_2^\mu/A$,

$$m^\mu, \bar{m}^\mu = (\delta_0^\mu \pm \mathrm{i}\delta_3^\mu/\sqrt{A})/\sqrt{2}, \tag{2.15}$$

(G_4 VI$_4$) $l^\mu = (\delta_0^\mu + \delta_1^\mu/A)/\sqrt{2}$, $n^\mu = (\delta_0^\mu - \delta_1^\mu/A)/\sqrt{2}$,

$$m^\mu, \bar{m}^\mu = (\delta_2^\mu \pm \mathrm{i}\delta_3^\mu)/\sqrt{2}B, \tag{2.16}$$

(G_4 VII$_2$) $l^\mu = \delta_1^\mu$, $n^\mu = [(-B^2/2A^2)\delta_1^\mu + \delta_2^\mu]/A^2$,

$$m^\mu, \bar{m}^\mu = (1/\sqrt{2})\delta_0^\mu \pm \mathrm{i}(x^2\delta_2^\mu + \delta_3^\mu)/A \tag{2.17}$$

(G_4 VIII$_1$) $l^\mu = (\delta_0^\mu + \delta_1^\mu/A)/\sqrt{2}$, $n^\mu = (\delta_0^\mu - \delta_1^\mu/A)/\sqrt{2}$,

$$m^\mu, \bar{m}^\mu = (\delta_2^\mu \pm \mathrm{i}\,\mathrm{cosec}\, x^2\delta_3^\mu)/\sqrt{2}B, \tag{2.18}$$

where from now on $A = A(t)$, $B = B(t)$ and differentiation with respect to t will be denoted by a dot.

The corresponding one-forms θ^a can be determined from the relations

$$\mathrm{d}x^\mu = l^\mu\theta^0 + n^\mu\theta^1 - m^\mu\theta^2 - \bar{m}^\mu\theta^3. \tag{2.19}$$

In terms of θ^a the metric takes the form

$$\mathrm{d}s^2 = 2(\theta^0\theta^1 - \theta^2\theta^3). \tag{2.20}$$

Now, the two-forms

$$Z^1 = \theta^3 \wedge \theta^0, \qquad Z^2 = \theta^1 \wedge \theta^2, \qquad Z^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^0 - \theta^2 \wedge \theta^3), \tag{2.21}$$

form a basis for the space of self dual two-forms (Cahen *et al* 1967). The exterior differential of these basic two-forms is given by

$$\begin{aligned} \mathrm{d}Z^1 &= -\tfrac{1}{2}\sigma^3 \wedge Z^1 - \sigma_2 \wedge Z^3, \\ \mathrm{d}Z^2 &= \tfrac{1}{2}\sigma_3 \wedge Z^2 + \sigma_1 \wedge Z^3, \\ \mathrm{d}Z^3 &= \tfrac{1}{2}\sigma_1 \wedge Z^1 - \tfrac{1}{2}\sigma_2 \wedge Z^2, \end{aligned} \tag{2.22}$$

where σ_1, σ_2 and σ_3 are three complex one-forms.

Writing these three forms in terms of the basic one-forms θ^a one obtains:

$$\sigma_p = \sigma_{pa}\theta^a, \qquad p = 1, 2, 3. \tag{2.23}$$

The quantities σ_{pa} are called the Ricci rotation coefficients which are equivalent to the twelve Newman–Penrose spin coefficients (Newman and Penrose 1962). The relation between the NP spin coefficients and σ_{pa} is known (Israel 1970).

We give now the one-forms corresponding to each type of metric (2.1)–(2.9); then using (2.21), (2.22), (2.23) and the relations between the σ_{pa} and the NP spin coefficients (Israel 1970), we obtain, after somewhat lengthy calculations, the following results.

(G_4 I)
$$\theta^0 = \mathrm{d}x^1, \qquad \theta^1 = -A\,\mathrm{d}x^3,$$
$$\sqrt{2}\theta^2 = -[\mathrm{d}t + \mathrm{i}B(\mathrm{d}x^2 + x^1\,\mathrm{d}x^3)], \qquad \theta^3 = \bar{\theta}^2, \tag{2.24}$$

and

$$\kappa = \sigma = \rho = \epsilon = \lambda = \mu = \nu = \gamma = 0,$$
$$\tau = -\pi = [(A/A) - \mathrm{i}(B/A)]/2\sqrt{2},$$
$$\alpha = [(-A/A) + 2(\dot{B}/B) + \mathrm{i}(B/A)]/4\sqrt{2},$$
$$\beta = -[(A/A) + 2(\dot{B}/B) + \mathrm{i}(B/A)]/4\sqrt{2}. \tag{2.25}$$

(G_4 III)
$$\sqrt{2}\theta^0 = \mathrm{d}t + B(\mathrm{d}x^2 + x^1\,\mathrm{d}x^3), \qquad \sqrt{2}\theta^1 = \mathrm{d}t - B(\mathrm{d}x^2 + x^1\,\mathrm{d}x^3),$$
$$\sqrt{2}\theta^2 = -A(\mathrm{d}x^1 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2 \tag{2.26}$$

and

$$\kappa = \sigma = \pi = \tau = \alpha = \beta = \lambda = \nu = 0,$$
$$\epsilon = -\gamma = -\tfrac{1}{4}[\sqrt{2}(\dot{B}/B) - \mathrm{i}(B/A)/\sqrt{2}],$$
$$\rho = -\mu = \tfrac{1}{2}[\sqrt{2}(\dot{A}/A) + \mathrm{i}(B/A)]. \tag{2.27}$$

(G_4 IV)
$$\theta^0 = \mathrm{d}x^2, \qquad \theta^1 = -B\mathrm{e}^{-x^1}\,\mathrm{d}x^3, \qquad \sqrt{2}\theta^2 = -(\mathrm{d}t + \mathrm{i}A\,\mathrm{d}x^1),$$
$$\theta^3 = \bar{\theta}^2 \tag{2.28}$$

and

$$\kappa = \sigma = \rho = \epsilon = \lambda = \mu = \nu = \gamma = 0,$$
$$\tau = -\pi = [(B/B) + \mathrm{i}/A]/2\sqrt{2},$$
$$\alpha = [2(A/A) - (\dot{B}/B) + \mathrm{i}/A]/4\sqrt{2},$$
$$\beta = -[2(A/A) + (\dot{B}/B) + \mathrm{i}/A]/4\sqrt{2}. \tag{2.29}$$

(G_4 V)
$$\sqrt{2}\theta^0 = \mathrm{d}t + A\,\mathrm{d}x^1, \qquad \sqrt{2}\theta^1 = \mathrm{d}t - A\,\mathrm{d}x^1,$$
$$\sqrt{2}\theta^2 = -B\mathrm{e}^{-x^1}(\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.30}$$

and

$$\kappa = \sigma = \pi = \tau = \alpha = \beta = \lambda = \nu = 0,$$
$$\epsilon = -\gamma = -(A/A)/4\sqrt{2}, \qquad \rho = -\mu = (B/B - 1/A)/\sqrt{2}. \tag{2.31}$$

(G_4 VI$_1$)
$$\theta^0 = \mathrm{d}x^2, \qquad \theta^1 = -B^2\,\mathrm{d}x^3,$$
$$\sqrt{2}\theta^2 = -(\mathrm{d}t + \mathrm{i}A\,\mathrm{d}x^1), \qquad \theta^3 = \bar{\theta}^2, \tag{2.32}$$

and

$$\kappa = \sigma = \rho = \epsilon = \lambda = \mu = \nu = \gamma = 0,$$
$$\tau = -\pi = B/\sqrt{2}B, \qquad \alpha = [A/A - \dot{B}/B]/2\sqrt{2}, \qquad \beta = -A/2\sqrt{2}A. \tag{2.33}$$

$(G_4\,\mathrm{VI}_3)$ $$\theta^0 = \mathrm{d}x^1 + (B/2A)\,\mathrm{d}x^2, \qquad \theta^1 = -A\,\mathrm{d}x^2,$$
$$\sqrt{2}\theta^2 = -(\mathrm{d}t + \mathrm{i}A\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.34}$$

and

$$\kappa = \sigma = \rho = \epsilon = \alpha = \lambda = \mu = \gamma = 0,$$
$$\beta = \pi = -\tau = -\dot{A}/2\sqrt{2}A, \qquad \nu = (1/2\sqrt{2}A)(B/A)^{\cdot}. \tag{2.35}$$

$(G_4\,\mathrm{VI}_4)$ $$\sqrt{2}\theta^0 = \mathrm{d}t + A\,\mathrm{d}x^1, \qquad \sqrt{2}\theta^1 = \mathrm{d}t - A\,\mathrm{d}x^1,$$
$$\sqrt{2}\theta^2 = -B(\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.36}$$

and

$$\kappa = \sigma = \pi = \tau = \lambda = \alpha = \beta = \nu = 0,$$
$$\epsilon = -\gamma = -\dot{A}/2\sqrt{2}A, \qquad \rho = -\mu = \dot{B}/\sqrt{2}B.$$

$(G_4\,\mathrm{VII}_2)$ $$\theta^0 = \mathrm{d}x^1 + (B^2/2A^2)(\mathrm{d}x^2 - x^2\,\mathrm{d}x^3), \qquad \theta^1 = A^2(\mathrm{d}x^2 - x^2\,\mathrm{d}x^3),$$
$$\theta^2 = -\tfrac{1}{2}(\sqrt{2}\,\mathrm{d}t + \mathrm{i}A\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.37}$$

and

$$\kappa = \sigma = \rho = \epsilon = \lambda = \mu = \gamma = 0,$$
$$\alpha = \mathrm{i}/4A, \qquad \beta = -\tfrac{1}{4}(2\sqrt{2}\dot{A}/A + \mathrm{i}/A), \tag{2.38}$$
$$\tau = -\bar{\pi} = \tfrac{1}{2}[\sqrt{2}(\dot{A}/A) + \mathrm{i}/A],$$
$$\nu = (-1/2A^2)[(B^2/\sqrt{2}A^2)^{\cdot} + \mathrm{i}B^2/A^3].$$

$(G_4\,\mathrm{VIII}_1)$ $$\sqrt{2}\theta^0 = \mathrm{d}t + A\,\mathrm{d}x^1, \qquad \sqrt{2}\theta^1 = \mathrm{d}t - A\,\mathrm{d}x^1,$$
$$\sqrt{2}\theta^2 = -B(\mathrm{d}x^2 + \mathrm{i}\sin x^2\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.39}$$

and

$$\kappa = \sigma = \pi = \tau = \lambda = \nu = 0,$$
$$\epsilon = -\gamma = -\dot{A}/2\sqrt{2}A, \qquad \alpha = -\beta = (1/2\sqrt{2}B)\cot x^2,$$
$$\rho = -\mu = \dot{B}/\sqrt{2}B. \tag{2.40}$$

From these results, we obtain the following geometrical properties for the null tetrad used here (Sachs 1961, Israel 1970).

(1) For all the above metrics, the congruences with tangent vectors l^μ are shear-free null geodesics.

(2) For the metrics G_4 I, IV, VI_3 and VII_2, the shear-free null geodesic congruence is expansion- and twist-free and parametrised by an affine parameter.

(3) For G_4 III the congruence of null geodesics has non-zero expansion and non-zero twist, while in G_4 V, VI_4 and VIII_1 the congruence has non-zero expansion but is twist-free.

3. Newman–Penrose equations and the Plebanski type of the Ricci tensor

Using the results of § 2, we can now insert the calculated spin coefficients for each Bianchi type of metrics (2.1)–(2.9) in the NP equations (Newman and Penrose 1962, Pirani 1965). We write the intrinsic derivatives occuring in the NP equations as follows:

$$D\varphi = \varphi_{;\mu} l^{\mu}, \qquad \Delta\varphi = \varphi_{;\mu} n^{\mu}, \qquad \delta\varphi = \varphi_{;\mu} m^{\mu}, \qquad \bar{\delta}\varphi = \varphi_{;\mu} \bar{m}^{\mu}. \tag{3.1}$$

From the NP equations we obtain, for each Bianchi type, the following results.

(G_4 I)

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad \phi_{00} = \phi_{01} = \phi_{12} = \phi_{22} = 0,$$

$$\delta\tau = \tau\bar{\tau} + \tau(\bar{\alpha} - \beta) + \Psi_2 + 2\Lambda,$$

$$(\tau - \bar{\tau})(\alpha - \beta) - \tau^2 + \Psi_2 - \Lambda + \phi_{11} = 0,$$

$$\delta\tau = \tau^2 - (\alpha - \bar{\beta})\tau - \phi_{20} = \tau^2 + (\beta - \bar{\alpha})\tau + \phi_{02},$$

$$\delta(\alpha - \beta) = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \Psi_2 + \Lambda + \phi_{11}. .$$

Since $\phi_{20} = \bar{\phi}_{02}$ and ϕ_{11} is real, we conclude from the above equations that B/A must be constant, i.e. $B = KA$, where K is a non-zero constant. So we have

$$3\Psi_2 = \tfrac{1}{4}[-(\dot{A}/A)^{\cdot} + (\dot{A}/A)^2 - 2K^2 + 3iK(A/A)],$$

$$6\Lambda = (A/A)^{\cdot} - \tfrac{11}{8}(\dot{A}/A)^2 + \tfrac{1}{8}K^2,$$

$$\phi_{11} = \tfrac{1}{4}\{(\dot{A}/A)^{\cdot} - \tfrac{3}{4}[K^2 + (\dot{A}/A)^2]\},$$

$$\phi_{02} = \tfrac{1}{4}\{(\dot{A}/A)^{\cdot} + \tfrac{1}{2}[K^2 + (\dot{A}/A)^2]\}.$$

Thus the metric of G_4 I is of Petrov type D. After a comparison with LS (Ludwig and Scanlan 1971, Hall 1976, Crade and Hall 1979), we see that the Ricci tensor has the Plebanski type $[T - 3S]_{[1-1]}$ with Segre characteristic [1(111)].

The Ricci tensor can have the degenerate Plebanski type $[2T - 2S]_{[1-1]}$ only if $\phi_{02} = 0$, which gives

$$A = B/K = K_2 \cos^2 \tfrac{1}{2}K(K_1 - t),$$

where K_1 and K_2 are constants of integration.

(G_4 III)

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad \phi_{01} = \phi_{02} = \phi_{12} = 0,$$

$$2D\gamma = 2\gamma(\gamma + \bar{\gamma}) + \Psi_2 - \Lambda + \phi_{11},$$

$$2\gamma(\rho - \bar{\rho}) - \rho^2 - \Psi_2 + \Lambda + \phi_{11} = 0,$$

$$\Psi_2 + 2\Lambda = -D\rho + \rho\bar{\rho} + \rho(\gamma + \bar{\gamma}),$$

$$\phi_{00} = \phi_{22} = D\rho - \rho^2 + \rho(\gamma + \bar{\gamma}).$$

Since ϕ_{00}, and ϕ_{22} are real, the last two above equations give $B^2 = KA^3$, where K is a non-zero constant. The other equations will be as follows:

$$3\Psi_2 = \tfrac{1}{4}[(\dot{A}/A)^{\cdot} - (\dot{A}/A)^2 + \tfrac{1}{2}KA + (3i/4)\dot{A}(K/\sqrt{A})],$$

$$24\Lambda = 7(A/A)^{\cdot} + \tfrac{33}{2}(\dot{A}/A)^2 - \tfrac{1}{2}KA,$$

$$\phi_{00} = \phi_{22} = \tfrac{1}{8}[4(\dot{A}/A)^{\cdot} + 2(\dot{A}/A)^2 + KA],$$

$$16\phi_{11} = 6(A/A)^{\cdot} - 5(\dot{A}/A)^2 - 3KA.$$

From these results, we see that the metric G_4 III is of Petrov type D and a comparison of LS shows that the Ricci tensor has the Plebanski type $[T - 2S_1 - S_2]_{[1-1-1]}$ with Segre characteristic [11(11)] but degeneracies are possible.

The Ricci tensor can have two double eigenvalues if $\phi_{00}=\phi_{22}=0$. In this case we obtain the function A as a solution of the differential equation

$$2AA-\dot{A}^2+\tfrac{1}{2}KA^3=0.$$

$(G_4\,\mathrm{IV})$

$$\Psi_0=\Psi_1=\Psi_3=\Psi_4=0, \qquad \phi_{00}=\phi_{01}=\phi_{12}=\phi_{22}=0,$$

$$\tau\bar{\tau}=\Psi_2-\Lambda+\phi_{11},$$

$$\delta(\alpha-\beta)=\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta-\Psi_2+\Lambda+\phi_{11},$$

$$\Psi_2+2\Lambda=\delta\tau+(\bar{\beta}-\alpha-\bar{\tau})\tau,$$

$$\phi_{02}=\delta\tau+(\bar{\alpha}-\beta-\tau)\tau.$$

Because of the reality condition for ϕ_{11}, we deduce that A must be constant, i.e. $A=1/K$ where K is a non-zero constant. With this condition, we obtain from the above equations the following results:

$$\Psi_2=\tfrac{1}{8}(\dot{B}/B)^{\cdot}, \qquad 16\Lambda=(\dot{B}/B)^{\cdot}-(\dot{B}/B)^2-K^2,$$

$$\phi_{02}=\bar{\phi}_{20}=\tfrac{1}{8}[2(\dot{B}/B)^{\cdot}-(\dot{B}/B)^2+K^2-2\mathrm{i}K(\dot{B}/B)],$$

$$16\phi_{11}=(B/B)^2+K^2.$$

So, the metric G_4 IV is of Petrov type D or O and a comparison with LS and Crade and Hall (1979) shows that the Ricci tensor has the Plebanski type $[T-3S]_{[1-1]}$ with Segre characteristic $[1(111)]$.

Now if $\Psi_2=0$, then the metric will be conformally flat. In this case, we have either $B=\text{constant}\neq 0$ or $B=K_2\,\mathrm{e}^{K_1 t}$. From LS and Crade and Hall (1979) we see, in each case, that the Ricci tensor will have also the Plebanski type $[T-3S]_{[1-1]}$ with Segre characteristic $[1(111)]$. In the case when $B=\text{constant}\neq 0$, the metric can be written in the form

$$-\mathrm{d}s^2=\mathrm{d}t^2+\mathrm{d}x^{1^2}+2\,\mathrm{e}^{x^1}\,\mathrm{d}x^2\,\mathrm{d}x^3,$$

with the transformation

$$\tilde{t}=2\,\mathrm{e}^{-x^1/2}\sin\tfrac{1}{2}t, \qquad \tilde{x}^1=2\,\mathrm{e}^{-x^1/2}\cos\tfrac{1}{2}t, \qquad \tilde{x}^2=x^2, \qquad \tilde{x}^3=x^3.$$

The metric now takes the form

$$-\mathrm{d}s^2=4(\tilde{t}^2+\tilde{x}^{1^2})^{-1}[\mathrm{d}\tilde{t}^2+\mathrm{d}\tilde{x}^{1^2}+2\,\mathrm{d}\tilde{x}^2\,\mathrm{d}\tilde{x}^3).$$

In the second case when $B=K_2\,\mathrm{e}^{K_1 t}$, we perform the coordinate transformation

$$\tilde{t}=\sqrt{2}\,\mathrm{e}^{-(t+x^1)/2}\sin\tfrac{1}{2}(t-x^1), \qquad \tilde{x}^1=\sqrt{2}\,\mathrm{e}^{-(t+x^1)/2}\cos\tfrac{1}{2}(t-x^1),$$

$$\tilde{x}^2=x^2, \qquad \tilde{x}^3=x^3,$$

so the metric takes the conformal form

$$-\mathrm{d}s^2=2(\tilde{t}^2+\tilde{t}^{1^2})^{-1}(\mathrm{d}\tilde{t}^2+\mathrm{d}\tilde{x}^{1^2}+2\,\mathrm{d}\tilde{x}^2\,\mathrm{d}\tilde{x}^3).$$

$(G_4\,\mathrm{V})$

$$\Psi_0=\Psi_1=\Psi_3=\Psi_4=0, \qquad \phi_{01}=\phi_{02}=\phi_{12}=0,$$

$$D\mu=\rho\mu+2\gamma\mu+\Psi_2+2\Lambda,$$

$$\Delta\rho=-\rho\mu+2\gamma\rho-\Psi_2-2\Lambda.$$

These two equations lead to the condition that A must be constant, so we can take $A=1/K(K\neq 0)$. Thus we have

$$6\Psi_2=-(\dot B/B)^{\cdot},\qquad 12\Lambda=-(\dot B/B)^{\cdot}+3[(\dot B/B)^2-K^2],$$
$$2\phi_{00}=(\dot B/B)^{\cdot}-[(\dot B/B)+K]^2,\qquad 4\phi_{11}=(\dot B/B)^2-K^2,$$
$$2\phi_{22}=(\dot B/B)^{\cdot}-[(\dot B/B)-K]^2.$$

Thus we conclude that the metric G_4V is of Petrov type D or O and after a comparison with LS, we see that the Ricci tensor has the Plebanski type $[T-2S_1-S_2]_{[1-1-1]}$ with Segre characteristic [11(11)] (degeneracies are possible) or the Plebanski type $[T-3S]_{[1-1]}$ with Segre characteristic [1(111)].

The Ricci tensor can have a null eigenvector if $\phi_{00}=0$ (or $\phi_{22}=0$).

Now, when $\phi_{00}=0$, then we have $B=[K_2/(K_1+t)]\,e^{-Kt}$, where K_1 and K_2 are constants of integration. In this case the Ricci tensor will have the Plebanski type $[2N-2S]_{[1-1]}$ with Segre characteristic [2(11)].

If $\phi_{11}=0$, we get $B=K_3\,e^{\pm Kt}$. For the positive sign we have $\phi_{00}=-K^2$, $\phi_{22}=0$, and for the negative sign we have $\phi_{00}=0$, $\phi_{22}=-K^2$. In either case we get $\Psi_2=0$, $\Lambda=0$ and a comparison with LS shows that the metric is of Plebanski type $[4N]_{[2]}$ with Segre characteristic [(211)]. The metric can be written in the form

$$ds^2=dt^2-dx^{1^2}-e^{2(x^1-t)}(dx^{2^2}+dx^{3^2}).$$

Using the coordinate transformation

$$\tilde t=e^{2(t-x^1)},\qquad 4\tilde x^1=t+x^1,\qquad \tilde x^2=x^2,\qquad \tilde x^3=x^3,$$

we can write the above metric in its conformal form

$$ds^2=\tilde t^{-1}(2d\tilde t\,d\tilde x^1-d\tilde x^{2^2}-d\tilde x^{3^2}).$$

(G_4 VI$_1$)
$$\Psi_0=\Psi_1=\Psi_3=\Psi_4=0,\qquad \phi_{00}=\phi_{01}=\phi_{12}=\phi_{22}=0,$$
$$3\Psi_2=-\tfrac12(\dot A/A)^{\cdot}+\tfrac54(\dot B/B)^{\cdot}+\tfrac12(\dot A/A)^2+\tfrac98(\dot B/B)^2+\tfrac12(\dot A/A)(\dot B/B),$$
$$6\Lambda=\tfrac12(\dot A/A)^{\cdot}+\tfrac74(\dot B/B)^{\cdot}-\tfrac12(\dot A/A)^2-\tfrac38(\dot B/B)^2+\tfrac52(\dot A/A)(\dot B/B),$$
$$4\phi_{11}=(\dot A/A)^{\cdot}-\tfrac12(\dot B/B)^{\cdot}-(\dot A/A)^2+\tfrac34(\dot B/B)^2+\dot A\dot B/AB,$$
$$2\phi_{02}=2\bar\phi_{20}=(\dot B/B)^{\cdot}-\tfrac32(\dot B/B)^2+\dot A\dot B/AB.$$

Thus the metric G_4 VI$_1$ is of Petrov type D or O and after a comparison with LS, we see that the Ricci tensor has the Plebanski type $[2T-S_1-S_2]_{[1-1-1]}$ with Segre characteristic [(11)11]. The Ricci tensor can have two double eigenvalues if $\phi_{02}=0$, which leads to the following relation between A and B:

$$B^{-3/2}=-\tfrac32K\int dt/A,\ \text{where } K \text{ is a non-zero constant.}$$

In this case the Ricci tensor will have the degenerate Plebanski type $[2T-2S]_{[1-1]}$ with Segre characteristic [(11)(11)].

(G_4 VI$_3$)
$$\Psi_0=\Psi_1=\Psi_2=\Psi_3=0,\qquad \phi_{00}=\phi_{01}=\phi_{12}=0,$$
$$4\Psi_4=[(1/A)(\dot B/A)^{\cdot}]^{\cdot}-\tfrac12(\dot A/A^2)(\dot B/A)^{\cdot}],$$

$$8\Lambda = (A/A)^{\cdot} - (\dot{A}/A)^2,$$

$$2\phi_{11} = \phi_{02} = \tfrac{1}{4}(\dot{A}/A)^{\cdot}, \qquad 4\phi_{22} = [(1/A)(B/A)^{\cdot}]^{\cdot} - \tfrac{5}{2}(\dot{A}/A^2)(B/A)^{\cdot}.$$

From these results, we see that the metric is of Petrov type N and a comparison with LS shows that the Ricci tensor has the Plebanski type $[3N-S]_{[2-1]}$ with Segre characteristic [(21)1]. The metric can be of type O if $\Psi_4 = 0$, from which we get $B = K_1 A \int A^{3/2}\,dt + K_2 A$.

Now, if $\Psi_4 = 0$ and $\phi_{11} = \phi_{02} = 0$, then the metric will be conformally flat and the Ricci tensor will have the degenerate Plebanski type $[4N]_{[2]}$ with Segre characteristic [(211)]. The vanishing of ϕ_{11} and ϕ_{02} give $A = K_1\,e^{Kt}$, where K is a non-zero constant and K_1 is a constant of integration. From $\Psi_4 = 0$, we have the solution

$$B = C_1\,e^{5Kt/2} + C_2\,e^{Kt},$$

where C_1 and C_2 are some constants,

$$(G_4\,\mathrm{VI}_4) \qquad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad \phi_{01} = \phi_{02} = \phi_{12} = 0,$$

$$6\Psi_2 = (A/A)^{\cdot} - (\dot{B}/B)^{\cdot} - (\dot{A}/A)^2 + (\dot{A}/A)(\dot{B}/B),$$

$$6\Lambda = -(B/B)^{\cdot} - 2(\dot{B}/B)^2 + (\dot{A}/A)(\dot{B}/B),$$

$$2\phi_{00} = 2\phi_{22} = (B/B)^{\cdot} + (\dot{B}/B)^2 + (\dot{A}/A)(\dot{B}/B),$$

$$4\phi_{11} = (A/A)^{\cdot} - (\dot{A}/A)^2 + (\dot{B}/B)^2.$$

From these results, we conclude that the metric is of Petrov type D or O. After a comparison with LS, we see that the Ricci tensor has the Plebanski type $[T-2S_1-S_2]_{[1-1-1]}$ with Segre characteristic [11(11)].

Now, a possible degeneracy of this Plebanski type can occur when A = constant $K \neq 0$ and $B = K_1\,e^{Kt}$. This will represent a space-like dust solution (Ray and Zimmerman 1977). In this case, the metric will be conformally flat and $\Lambda \neq 0$. The Ricci tensor will have the Plebanski type $[T-3S]_{[1-1]}$ with Segre characteristic [1(111)].

Using the transformation

$$K\tilde{t} = e^{-Kt}\cosh K^2 x^1, \qquad K\tilde{x}^1 = e^{-Kt}\sinh K^2 x^1, \qquad \tilde{x}^2 = K_1 x^2, \qquad \tilde{x}^3 = K_1 x^3,$$

we can write the metric in its conformal form

$$ds^2 = (\tilde{t}^2 - \tilde{x}^{12})^{-1}(d\tilde{t}^2 - d\tilde{x}^{12} - d\tilde{x}^{22} - d\tilde{x}^{32}).$$

Another degeneracy of the Plebanski type can also occur when $A = B = K_2\,e^{Kt}$. This will represent a conformally flat de Sitter solution (Ray and Zimmerman 1977) with $\Lambda \neq 0$.

Using the transformation

$$K\tilde{t} = e^{-Kt}, \qquad \tilde{x}^1 = K_2 x^1, \qquad \tilde{x}^2 = K_2 x^2, \qquad \tilde{x}^3 = K_2 x,$$

we can write the metric in its conformal form

$$ds^2 = K^{-2}\tilde{t}^{-2}(d\tilde{t}^2 - d\tilde{x}^{12} - d\tilde{x}^{22} - d\tilde{x}^{32}).$$

Finally, if $\Lambda = 0$ and $\phi_{00} = \phi_{22} = 0$, then we get $A = K_3(C_1 t + C_2)^{1/3}$ and $B = (C_1 t + C_2)^{2/3}$, where K_3, C_1 and C_2 are constants of integration. In this case, we see that $\phi_{11} = 0$. So, we have the Kasner vacuum solution (Ray and Zimmerman 1977). The

Ricci tensor has the most degenerate type $[4T]_{[1]}$ with Segre characteristic [(1111)].

$(G_4\,\mathrm{VII}_2)$ $\Psi_0=\Psi_1=\Psi_3=0, \qquad \phi_{00}=\phi_{01}=\phi_{12}=0,$

$$\pi\bar{\pi}=\Psi_2-\Lambda+\phi_{11}, \tag{i}$$

$$\delta\alpha-\delta\beta=\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta-\Psi_2+\Lambda+\phi_{11}, \tag{ii}$$

$$-\delta\pi=\pi\bar{\pi}-(\bar{\alpha}-\beta)\pi+\Psi_2+2\Lambda, \tag{iii}$$

$$-\delta\bar{\pi}=\pi\bar{\pi}+(\bar{\beta}-\alpha)\bar{\pi}+\Psi_2+2\Lambda, \tag{iv}$$

$$-\delta\nu=(3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu-\Psi_4, \tag{v}$$

$$\phi_{02}=\bar{\phi}_{20}=-\delta\bar{\pi}-\bar{\pi}^2-(\alpha-\bar{\beta})\bar{\pi}, \tag{vi}$$

$$\delta\nu=-\pi\bar{\nu}-(3\beta+\bar{\alpha}+\bar{\pi})\nu+\phi_{22}. \tag{vii}$$

Now, from equations (iii) and (iv) we see that A must be constant. With $A=K\neq 0$, we obtain from equations (i)–(iv), $\Psi_2=0$ and $\Lambda=-1/8K^2$. From equations (v), (vi) and (vii) we obtain the following results:

$$\Psi_4=-(1/4K^4)[(B^2)^{\cdot\cdot}-2(B^2)/K+2\sqrt{2}\mathrm{i}B^2/K^2],$$

$$\phi_{02}=-2\Lambda, \qquad \phi_{11}=-\Lambda,$$

$$\phi_{22}=-(1/2K^4)(BB+\dot{B}^2+2B^2/K^2).$$

From these results and the comments about the Petrov types given in the Introduction we see that the metric $G_4\,\mathrm{VII}_2$ must be of type O, i.e. $\Psi_4=0$; consequently $B=0$ and in this case the metric does not have Lorentz signature.

Now if $\phi_{02}=0$, we obtain the solution $B^2=K_1\sin(2t/K)+K_2\cos(2t/K)$. In this case the Ricci tensor will have the Plebanski type $[3T-S]_{[1-1]}$.

$(G_4\,\mathrm{VIII}_1)$ $\Psi_0=\Psi_1=\Psi_3=\Psi_4=0, \qquad \phi_{01}=\phi_{02}=\phi_{12}=0,$

$$D\alpha=\rho\alpha, \tag{i}$$

$$D\beta=-\rho\beta, \tag{ii}$$

$$\Delta\beta=\rho\beta, \tag{iii}$$

$$D\gamma-\Delta\epsilon=4\epsilon^2+\Psi^2-\Lambda+\phi_{11}, \tag{iv}$$

$$\delta\alpha-\bar{\delta}\beta=-\rho^2+4\alpha^2-\Psi_2+\Lambda+\phi_{11}, \tag{v}$$

$$\Delta\alpha=\rho^2-2\epsilon\rho-\Psi_2-2\Lambda, \tag{vi}$$

$$D\rho=\rho^2+2\rho\epsilon+\phi_{00}, \tag{vii}$$

$$-\Delta\mu=\rho^2+2\rho\epsilon+\phi_{22}. \tag{viii}$$

Now, equations (i), (ii) and (iii) can only be satisfied if $B=$ constant. With $B=1/K$, where K is a non-zero constant, we obtain from the above equations the following results:

$$6\Psi_2=-12\Lambda=(\dot{A}/A)^{\cdot}-(\dot{A}/A)^2+K^2+2K^2\cot^2x^2,$$

$$4\phi_{11}=(A/A)^{\cdot}-(\dot{A}/A)^2-K^2-2K^2\cot^2x^2,$$

$$\phi_{00}=\phi_{22}=0.$$

Thus the metric is of Petrov type D. After a comparison with LS, we see that the Ricci tensor has the Plebanski type $[2T-2S]_{[1-1]}$ with Segre characteristic $[(11)(11)]$.

Acknowledgments

The author would like to express his gratitude to Dr G S Hall for attracting the author's attention to the paper of Ray and Zimmerman and to Dr Hall and Dr M A H MacCallum for sending him copies of their papers prior to publication. The author would like also to acknowledge the valuable comments of the referees.

References

Barnes A 1974 *Gen. Rel. Grav.* **5** 149
Cahen M, Debever R and Defrise L 1967 *J. Math. Mech.* **16** 761
Crade R F and Hall G S 1979 *Phys. Lett.* **75A** 17
Ehlers and Kundt 1962 in *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley)
Eisenhart L 1961 *Continuous Groups of Transformations* (New York: Dover)
Goenner H and Stachel J 1970 *J. Math. Phys.* **11** 3358
Hall G S 1976 *J. Phys. A: Math. Gen.* **9** 541
Israel W 1970 *Commun. Dublin Inst. Adv. Studies Ser. A.* No. 19
Ludwig W and Scanlan G 1971 *Commun. Math. Phys.* **20** 291
MacCallum M A H 1973 in *Cargése Lectures in Physics* ed Schatzmann (New York: Gordon and Breach) **6** 61–174
—— 1979 *Preprint* Queen Mary College, University of London
Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566
Petrov A Z 1969 *Einstein spaces* (New York: Pergamon)
Pirani F 1965 in *Brandeis Summer Inst. Theor. Phys.* vol 1 (Englewood Cliffs, New York: Prentice-Hall)
Plebanski J 1964 *Acta. Phys. Pol.* **26** 963
Plebanski J and Stachel J 1968 *J. Math. Phys.* **9** 269
Ray J R and Zimmerman J Ch 1977 *J. Math. Phys.* **18** 881
Sachs R K 1961 *Proc. R. Soc.* A **264** 309